Problem 2) Equating the first-order derivatives of $g(x, y) = a + bx + cy + dx^2 + ey^2 + fxy$ with respect to x and y to zero, we find the point (x_0, y_0) that is the location of either a minimum, or a maximum, or a saddle point, as follows:

$$\begin{cases} \dfrac{\partial g}{\partial x} = b + 2dx_0 + fy_0 = 0 \\ \dfrac{\partial g}{\partial y} = c + 2ey_0 + fx_0 = 0 \end{cases} \rightarrow \quad \begin{pmatrix} 2d & f \\ f & 2e \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = -\begin{pmatrix} b \\ c \end{pmatrix}$$

$$\rightarrow \quad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = -\begin{pmatrix} 2d & f \\ f & 2e \end{pmatrix}^{-1} \begin{pmatrix} b \\ c \end{pmatrix} = -\frac{1}{4de - f^2} \begin{pmatrix} 2e & -f \\ -f & 2d \end{pmatrix} \begin{pmatrix} b \\ c \end{pmatrix}$$

$$\rightarrow \quad \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = \begin{pmatrix} \dfrac{2eb - fc}{f^2 - 4de} \\ \dfrac{2dc - fb}{f^2 - 4de} \end{pmatrix}.$$

← point at which the function $g(x, y)$ is either a maximum, a minimum, or a saddle point.

The second derivatives of $g(x, y)$ are now evaluated as follows:

$$g_{xx} = \frac{\partial^2 g(x,y)}{\partial x^2} = 2d; \qquad g_{yy} = \frac{\partial^2 g(x,y)}{\partial y^2} = 2e; \qquad g_{xy} = \frac{\partial^2 g(x,y)}{\partial x \partial y} = f.$$

Normally, we would have to evaluate the second-order derivatives at (x_0, y_0). In this problem, however, the second-order derivatives are constants (i.e., independent of x and y). We thus proceed to write down the conditions for the existence of a maximum or a minimum.

$$g_{xy}^2 < g_{xx} g_{yy} \quad \rightarrow \quad f^2 < 4ed \quad \begin{matrix} \nearrow \text{ both } e \text{ and } d \text{ positive} \quad \rightarrow \quad g(x_0, y_0) \text{ is minimum.} \\ \searrow \text{ both } e \text{ and } d \text{ negative} \quad \rightarrow \quad g(x_0, y_0) \text{ is maximum.} \end{matrix}$$

If $f^2 > 4ed$, there will be two straight lines in the xy-plane that go through (x_0, y_0). These lines divide the xy-plane into four regions. In two of these regions $g(x, y) > g(x_0, y_0)$, while in the remaining two $g(x, y) < g(x_0, y_0)$. The point (x_0, y_0) is, therefore, a saddle point.

If $f^2 = 4ed$, the point (x_0, y_0) may not exist; see the denominator of the expressions obtained above for x_0 and y_0. In this case, $\partial g/\partial x = 0$ along one straight line, while $\partial g/\partial y = 0$ along another straight line. These two lines are parallel to each other and, therefore, do not cross. Since the loci of $\partial g/\partial x = 0$ and $\partial g/\partial y = 0$ do not have a common point, the point (x_0, y_0) does *not* exist in this case. If these two straight lines happen to overlap, however, then $g(x, y)$ will be constant along the entire line. Instead of a single minimum or maximum, there now exists an entire line over which $g(x, y)$ is either a minimum (if both e and d are positive), or a maximum (if both e and d are negative). The slope $\Delta y/\Delta x$ of this line is given by $-2d/f$.